An arrival-time distribution for the equilibrium mean waiting time of a discrete-time single-server queue with acceptance period and Poissonian population of customers

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Abstract We study a discrete-time first-come first-served (FCFS) single-server queue with an acceptance period and no early arrival. The number of arriving customers is Poisson distributed, and their service times are generally distributed. Customers choose their arrival times with the goal of minimizing their expected waiting times. In this study, we show an arrival-time distribution of customers for the equilibrium mean waiting time.

1. Introduction

Many real-life queueing systems have acceptance periods for arriving customers, that is, the system accepts arriving customers only during the period between the opening and closing times. Typical examples of such queueing systems are restaurant in lunch time, service counter at bank (or government office), and rush-hour congestion in transportation networks. Customers of these systems face with the decision problem of when to arrive at the systems so as to achieve a certain goal, e.g., to minimize the waiting time for service. Customers' decisions on arrival, naturally, interact with each other, and thus the arrival times of customers are endogenously determined.

Glazer and Hassin [1]'s work is a pioneer study on the decision problem on when to arrive at the queueing system with an acceptance period. Glazer and Hassin studied a continuous-time first-come first-served (FCFS) single-server queue with an acceptance period, where a Poisson-distributed number of homogeneous customers arrive at the system and customers may arrive at the system before its opening time, i.e., early arrival of customers is allowed. The service times are assumed to be independently and identically distributed according to an exponential distribution. Glazer and Hassin assumed that customers choose their arrival times with the goal of minimizing their expected waiting times. More specifically, they studied the queueing model by a non-cooperative game with a random number of players, and obtained an equilibrium strategy of arriving customers as a mixed strategy.

Similar queueing systems have been studied. Hassin and Kleiner [2] studied a queueing system without early arrival, and obtained an equilibrium strategy of arriving customers. Hassin and Kleiner [2] also reported that no early arrivals reduce the mean waiting time in equilibrium especially when the system is heavily loaded (see Figure 3 therein). Haviv [3] and Ravner [4] studied queueing models with a tardiness cost and an arrival order cost, respectively, as well as waiting cost.

In this paper, we consider a discrete-time FCFS single-server queue with an acceptance period and no early arrivals, and discuss an arriving-time distribution achieving the equilibrium mean waiting time, which is referred to as the *equilibrium arriving-time distribution*. As in the preceding studies [1, 2, 3, 4], we assume that the number of arriving customers is Poisson distributed. However, our study has three differences from these preceding studies. First, our model is in discrete time whereas the existing models in the preceding studies are in continuous time. The assumption of discrete time facilitates the computation of an equilibrium mean waiting time and the corresponding equilibrium arriving-time distribution. Second, the service times in our model are generally distributed whereas those in the existing models are exponentially distributed. Thus, our model enables us to investigate the effect of the service time distribution on the arrival strategy of customers. Third, the waiting cost of our model is the mean *actual* waiting time whereas that of the existing models is the mean *virtual* waiting time.

The rest of the paper is organized as follows. Section 2 describes the queueing model studied in this paper, and provides some fundamental results on the mean workload and the actual waiting time. Section 3 introduces the notion of equilibrium mean waiting time and equilibrium arriving-time distribution, and then presents a procedure for calculating an equilibrium mean waiting time and the corresponding equilibrium arriving-time distribution.

2. Preliminaries

2.1. Model description

We consider a discrete-time first-come, first-served (FCFS) queueing system with infinite waiting room and one server as follows. The time axis of the system is divided into fixed-length time intervals, where each time interval is referred to as slot t for nonnegative integer $t \in \mathbb{Z}+ := \{0, 1, 2, \ldots\}$. Without loss of generality, each time slot is assumed to have length one. The system is open for arriving customers during an acceptance period, which is defined by a set of slots $\mathcal{T} := \{0, 1, 2, \ldots, T\}$ for a positive integer T. The arriving customers entering during the same time-slot are served in random order. We assume that the server is available to continue service until all customers who arrived during the acceptance period $\mathcal{T}$ are served.

In what follows, some probabilistic assumptions on our model are listed.

Assumption 2.1 (Population and arrival time distribution) The number of customers seeking service from the system is denoted by a Poisson random variable A with positive mean λ, and each arriving customers independently chooses its arrival time from the acceptance period $\mathcal{T}$ with a common probability distribution $\mathcal{P} := \{p_t; t \in \mathcal{T}\}$[1)].

Assumption 2.2 (Arrival instants of customers) In each slot $t \in \mathcal{T}$, the arrival of customers can occur immediately after the slot starts. Let A_t, $t \in \mathcal{T}$, denote the number of arrivals in slot t, then we have from Assumption 2.1, for $t \in \mathcal{T}$,

$$\mathsf{P}(A_t = n) = e^{-\lambda p_t} \frac{(\lambda p_t)^n}{n!}, \qquad n \in \mathbb{Z}_+. \tag{2.1}$$

Assumption 2.3 (Service requirement and processing instants) The service times of customers arriving at the system in slots 0 through T are independent and identically distributed (i.i.d.) with discrete distribution $\{b(k); k \in \mathbb{N}\}$ having finite positive mean $\bar{b}$. Thus, let B denote a generic random variable for the service time. It then follows that $\mathsf{P}(B = k) = b(k)$ for $k \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and

$$\bar{b} := \mathsf{E}[B] = \sum_{k=1}^{\infty} kb(k). \tag{2.2}$$

If the server holds a customer in the middle of slot t ($t \in \mathbb{Z}_+$), it processes the customer's service requirement by 1 at the end of the slot.

We note that the system is always stable because the acceptance period $\mathcal{T}$ is a finite set and the expected total workload into the system is finite, i.e., $\mathsf{E}[A]\bar{b} = \lambda\bar{b} < \infty$. For $t \in \mathcal{T}$, let $B_{t,i}$'s, $i \in \{1, 2, \ldots, A_t\}$, denote the service times of the customers arriving at slot t. For $t \in \mathcal{T}$, let X_t denote the total service times of the customers arriving at slot t, and $x_t(k) = \mathsf{P}(X_t = k)$ for $k \in \mathbb{N}$. It then follows from Assumption 2.2 and 2.3 that

$$X_t = \sum_{i=1}^{A_t} B_{t,i}, \qquad t \in \mathcal{T}. \tag{2.3}$$

1) The common arrival distribution $\mathcal{P}$ is referred to as a symmetric arrival strategy profile in Haviv 2013.

and thus

$$\sum_{k=0}^{\infty} z^k x_t(k) = \sum_{n=0}^{\infty} e^{-\lambda p_t} \frac{(\lambda p_t)^n}{n!} (\widetilde{b}(z))^n = \exp\left\{-\lambda p_t \left(1 - \widetilde{b}(z)\right)\right\}, \tag{2.4}$$

where

$$\widetilde{b}(z) = \sum_{k=1}^{\infty} z^k b(k).$$

Furthermore, since $\mathsf{E}[B] = \overline{b}$ and $\mathsf{E}[A_t] = \lambda p_t$ for $t \in \mathcal{T}$, we have

$$\mathsf{E}[X_t] = \mathsf{E}[A_t]\mathsf{E}[B] = \lambda p_t \overline{b}, \qquad t \in \mathcal{T}. \tag{2.5}$$

Finally, we assume the following.

Assumption 2.4 The system is empty at the beginning of slot 0, i.e., immediately before the arrivals of customers (if any) in slot 0.

2.2. Workload in system and mean waiting time

We first consider workload in system. Let V_{t-}, $t \in \mathcal{T}$, denote the total unfinished workload immediately before the beginning of slot t. Let $v_t(k) = \mathsf{P}(V_{t-} = k)$ for $t \in \mathcal{T}$ and $k \in \mathbb{Z}_+$. Assumption 2.4 shows that $V_{0-} = 0$, i.e., $v_0(0) = 1$, and by definition, we have

$$V_{t-} = \left(V_{(t-1)-} + X_{t-1} - 1\right)^+, \qquad t \in \{1, 2, \dots, T\}, \tag{2.6}$$

where $(x)^+ = \max(x, 0)$ for $x \in (-\infty, \infty)$. Thus, we obtain the following result.

Lemma 2.1 *We have $v_0(0) = 1$ and $v_0(k) = 0$ for $k \in \mathbb{N}$, and*

$$v_t(k) = \begin{cases} v_{t-1}(0)x_{t-1}(0) + v_{t-1}(0)x_{t-1}(1) + v_{t-1}(1)x_{t-1}(0), & k = 0, \\ \displaystyle\sum_{\ell=0}^{k+1} v_{t-1}(\ell)x_{t-1}(k+1-\ell), & k \in \mathbb{N}. \end{cases} \tag{2.7}$$

for $t = 1, 2, \dots, T$. Particularly, we have for $t = 1$,

$$v_1(k) = \begin{cases} x_0(0) + x_0(1), & k = 0, \\ x_0(k+1), & k \in \mathbb{N}, \end{cases} \tag{2.8}$$

Proof. Equation (2.7) is obvious by Assumptions 2.3, 2.4, and the recursion formula (2.6) because the unfinished workload (if any) is processed by 1 at the end of each slot. □

Lemma 2.2

$$\mathsf{E}[V_{1-}] = \lambda p_0 \overline{b} + 1 - e^{-\lambda p_0}, \tag{2.9}$$

$$\mathsf{E}[V_{t-}] = \mathsf{E}[V_{(t-1)-}] + \lambda p_{t-1}\overline{b} - (1 - e^{-\lambda p_{t-1}} v_{t-1}(0)), \qquad t = 2, 3, \dots, T. \tag{2.10}$$

Proof. From (2.6), we have for $t \in \mathcal{T}$,

$$\begin{aligned} \mathsf{E}[V_{t-}] &= \mathsf{E}[(V_{(t-1)-} + X_{t-1} - 1)\mathbb{I}(V_{(t-1)-} + X_{t-1} \ge 1)] \\ &= \mathsf{E}[V_{(t-1)-}] + \mathsf{E}[X_{t-1}] - (1 - \mathsf{P}(V_{(t-1)-} + X_{t-1} = 0)). \end{aligned} \tag{2.11}$$

Furthermore, we note that

$$\mathsf{P}(V_{(t-1)-} + X_{t-1} = 0) = \mathsf{P}(V_{(t-1)-} = 0)\mathsf{P}(X_{t-1} = 0) \tag{2.12}$$

because $V_{(t-1)-}$ and X_{t-1} are independent and nonnegative random variables. Equations (2.11) and (2.12) complete the proof. □

Next, we consider the (actual) waiting time of an arbitrary customer arriving in a slot. Let $\mathcal{T}^+ = \{t \in \mathcal{T}; p_t > 0\}$. We then define W_t, $t \in \mathcal{T}^+$, as the waiting time of an arbitrary customer arriving in slot t.

Lemma 2.3

$$\mathsf{E}[W_t] = \mathsf{E}[V_{t-}] + \frac{\overline{b}}{2}\left(\frac{\lambda p_t}{1-e^{-\lambda p_t}} - 1\right), \qquad t \in \mathcal{T}^+. \tag{2.13}$$

Proof. For $t \in \mathcal{T}^+$, let $q_t(k)$, $k \in \mathbb{Z}_+$, denote the probability that a customer randomly chosen from the ones arriving in slot t enters the server after the $k-1$ members of them receive service and leave the system. Recalling that the arriving customers during the same slot are randomly ordered, then it follows from (2.1) that

$$\begin{aligned} q_t(k) &= \sum_{\ell=k+1}^{\infty} \frac{1}{\ell}\frac{\mathsf{P}(A_t=\ell)}{\mathsf{P}(A_t \ge 1)} = \sum_{\ell=k+1}^{\infty} \frac{1}{\ell}\frac{\mathsf{P}(A_t=\ell)}{1-\mathsf{P}(A_t = 0)} \\ &= \sum_{\ell=k+1}^{\infty} \frac{1}{\ell}\frac{e^{-\lambda p_t}}{1-e^{-\lambda p_t}}\frac{(\lambda p_t)^\ell}{\ell!}, \qquad k \in \mathbb{Z}_+. \end{aligned} \tag{2.14}$$

Thus, for $t \in \mathcal{T}^+$, we have

$$\begin{aligned} \mathsf{E}[W_t] &= \mathsf{E}[V_{t-}] + \sum_{k=0}^{\infty} k q_t(k) \times \overline{b} \\ &= \mathsf{E}[V_{t-}] + \sum_{\ell=1}^{\infty} \frac{1}{\ell}\frac{e^{-\lambda p_t}}{1-e^{-\lambda p_t}}\frac{(\lambda p_t)^\ell}{\ell!}\sum_{k=0}^{\ell-1} k\overline{b} \\ &= \mathsf{E}[V_{t-}] + \frac{\overline{b}}{2}\sum_{\ell=1}^{\infty}(\ell-1)\frac{e^{-\lambda p_t}}{1-e^{-\lambda p_t}}\frac{(\lambda p_t)^\ell}{\ell!} \\ &= \mathsf{E}[V_{t-}] + \frac{\overline{b}}{2}\left(\frac{\lambda p_t}{1-e^{-\lambda p_t}} - 1\right), \end{aligned}$$

which shows that (2.13) holds. □

3. Arrival-Time Distribution for Equilibrium Mean Waiting Time

We introduce the *equilibrium mean waiting time* and *equilibrium arrival-time distribution*, and present a procedure for calculating them.

Definition 3.1 A positive value w^* is said to be an *equilibrium mean waiting time* if

$$\mathsf{E}[W_t] = w^*, \quad t \in \mathcal{T}^+, \tag{3.1}$$

$$\mathsf{E}[V_{t-}] > w^*, \quad t \in \mathcal{T} \setminus \mathcal{T}^+. \tag{3.2}$$

The arrival-time distribution $\mathcal{P} = \{p_t; t \in \mathcal{T}\}$ is said to be *equilibrium* if it ensures the existence of an equilibrium mean waiting time w^*.

Let $\mathscr{P}^*$ denote the set of equilibrium arrival-time distributions. Let $\mathcal{P}^* := \{p_t^*; t \in \mathcal{T}\}$ denote an arbitrarily element of $\mathscr{P}^*$. In the rest of this paper, we fix $\mathcal{P} = \mathcal{P}^*$, i.e., $p_t = p_t^*$ for all $t \in \mathcal{T}$. It follows from $V_{0-} = 0$, Definition 3.1 and Lemma 2.3 that $0 \in \mathcal{T}^+$, i.e.,

$$p_0^* > 0, \tag{3.3}$$

otherwise, i.e., if $0 \in \mathcal{T} \setminus \mathcal{T}^+$, we have $0 = \mathsf{E}[V_{0-}] > w^*$ by (3.2), which contradicts to $w^* > 0$. We then have

$$w^* = \mathsf{E}[W_0] = \frac{\overline{b}}{2}\left(\frac{\lambda p_0^*}{1-e^{-\lambda p_0^*}} - 1\right). \tag{3.4}$$

Remark 3.1 *It is easy to see that $x/(1-e^{-x})$ is increasing in $x \ge 0$ and*

$$\lim_{x\to 0} \frac{x}{1-e^{-x}} = 1. \tag{3.5}$$

Therefore, (3.4) implies that w^* is positive and increasing with $p_0^* \in (0,1)$.

It follows from Lemma 2.3, Definition 3.1 and (3.4) that p_t^* $(t = 1, 2, \ldots, T)$ satisfies following equation.

$$\begin{aligned}\frac{\bar{b}}{2}\left(\frac{\lambda p_t^*}{1-e^{-\lambda p_t^*}} - 1\right) &= (w^* - \mathsf{E}[V_{t-}])^+ \\ &= \frac{\bar{b}}{2}\left(\frac{\lambda p_0^*}{1-e^{-\lambda p_0^*}} - 1 - \frac{2\mathsf{E}[V_{t-}]}{\bar{b}}\right)^+, \qquad t = 1, 2, \ldots, T, \end{aligned}\tag{3.6}$$

which results in the following theorem.

Theorem 3.1 *For $t = 1, 2, \ldots, T$, the probability p_t^* is a solution of the following equation:*

$$\frac{\lambda p_t^*}{1-e^{-\lambda p_t^*}} = \left(\frac{\lambda p_0^*}{1-e^{-\lambda p_0^*}} - 1 - \frac{2\mathsf{E}[V_{t-}]}{\bar{b}}\right)^+ + 1, \qquad t = 1, 2, \ldots, T, \tag{3.7}$$

$$\mathsf{E}[V_{t-}] = \mathsf{E}[V_{(t-1)-}] + \lambda p_{t-1}^* \bar{b} - (1 - e^{-\lambda p_{t-1}^*} v_{t-1}(0)), \qquad t = 1, 2, \ldots, T, \tag{3.8}$$

$$\sum_{t\in\mathcal{T}} p_t^* = 1, \tag{3.9}$$

where the last equation is the normalizing condition for the probability distribution $\{p_t^*; t \in \mathcal{T}\}$.

Proof. Equations (3.7) and (3.8) are immediately obtained from (3.6) and (2.10). □

We close this section by summarizing the computational procedure for the equilibrium arrival-time distribution $\mathcal{P}^* = \{p_t^*; t \in \mathcal{T}\}$.

- Step 0: Set $p_0^* = \varepsilon$ for a small $\varepsilon > 0$, and choose $\delta > 0$ as a precision parameter.
- Step 1: Compute w^* by (3.4).
- Step 2: For $t = 1, 2, \ldots, T$, compute p_t^* by (3.7), where $\mathsf{E}[V_{t-}]$ and $v_{t-1}(0)$ are recursively computed by (3.8) and (2.7), respectively.
- Step 3: If $|1 - \sum_{t\in\mathcal{T}} p_t^*| < \delta$, return $\{p_t^*; t \in \mathcal{T}\}$ as the equilibrium solution, otherwise $p_0^* := p_0^* + \varepsilon$ and go to Step 1.

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A Finite Bottleneck Game with Homogeneous Commuters

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Abstract This paper investigates the finite bottleneck game, in which we assume a finite set of commuters and a finite set of departing time slots. We show that the set of Nash equilibria is equivalent to the set of strong Nash equilibria when we assume homogeneous commuters in their preferences. We also show that pure-strategy Nash equilibria do not exist in general in this setting. Moreover, when we allow commuters to differ in their preferences, we show that Nash equilibria may not exist, and the equivalence result no longer follows.

1. Introduction

A bottleneck model is used in analyzing a rush-hour traffic congestion, where commuters depart from their origins (e.g. their houses) to their destinations (e.g. their workplaces). The simplest model was independently analyzed by Vickrey (1969) and Hendrickson and Kocur (1981), where a continuum of commuters depart from a single origin to a single destination connected by a single road. Along the road, there is a bottleneck in which a queue forms if there are too many commuters in the bottleneck at a given time. In these papers, commuters decide on the departure time based on the trade-offs between congestion and their optimal arrival time.

Moreover, these studies assume homogeneous commuters in that all commuters have the same preferred time of arrival and a specific form of the trip cost function. Their contribution was that departure time decision made by commuters are endogenously determined by means of including trade-off between their travel time and their arrival time.

Other subsequent papers, such as Smith (1983), Daganzo (1985) and Arnott et al. (1990), also consider a continuum of commuters and a continuous time horizon. However, a finite set of commuters and a discrete time horizon seem closer to real-life situations in which the population of a city is finite. The situation corresponds to where there is a relatively small number of commuters, each of which can cause congestion to occur.

Our model, which we call the finite bottleneck game, is endowed with a finite set of commuters and a finite set of time periods, each of which is called a slot. Commuters have preference on two arguments: her departure time and the queue-length which she have to wait through the bottleneck, where in this model the capacity is the maximum number of commuters that can pass through it in each slot. We assume homogeneous commuters as in the previous studies above, but we do not give a specific form of trip costs function. In this sense, our model is an abstract generalization of models of the aforementioned papers.

Mathematically, our model is also an extension of the congestion game (c.f. Rosenthal (1973)). The congestion game considers a situation in which n players choose a combination of primary factors out of t alternatives. Each player's payoff is determined by the sum of the costs of each primary factor she chooses, while the cost of each primary factor depends on the number of players who choose it, and not on the players' names. Rosenthal (1973) proved that there always exists at least one pure-strategy Nash equilibrium by constructing a potential function, which is later formalized by Monderer and Shapley (1996).

Though Rosenthal (1973) and Monderer and Shapley (1996) assume that the cost functions, hence payoff functions, have the same form among the players who take same factors, Milchtaich (1996) allows payoff functions to be different between players in his model, where players choose only one factor from a common set of factors. In Milchtaich (1996), it was shown that a Nash equilibrium always exists in pure strategies. Moreover, Konishi et al. (1997a) shows that in the same model, the set of strong Nash equilibria, which is a stronger concept than Nash equilibria, is nonempty.

Specifically, Konishi et al. (1997a) describes the games in the above class using the following three properties: anonymity [A], partial rivalry [PR] and independence of irrelevant choices [IIC][1]. First, [A] requires that the payoff of each player depends on the number of players who choose each action and not

[1] [IIC] condition is also called no spillovers [NS] in Konishi et al. (1997b)

on the players' names. [PR] states that the payoff of each player increases if another player who had chosen the same strategy chooses a different strategy. Finally, [IIC] states that the payoff of a player is not affected even if another player that chooses a different strategy from hers switches to another strategy that is also different strategy from hers.

In relation to congestion games, our model does not satisfy [IIC], whereas the other two conditions hold. Specifically, [IIC] would be violated in the case where a player who had departed later then switched to an earlier departure time and thereby possibly creating a longer queue for some of those players which she leaps over.

In this paper, we restrict our attention to the special case where commuters are homogeneous. Then, we can show that a set of Nash equilibria coincides with that of strong Nash equilibria (Proposition 3.1), while pure-strategy Nash equilibria do not exist in general. It can be stated that this proposition partly explains the difficulty in the existence of a Nash equilibrium. In addition, we illustrate that the equivalence of the set of Nash equilibria and that of strong Nash equilibria does not hold when the homogeneity assumption is dropped.

The rest of the paper is organized as follows: in Section 2, we define the model and notations. In Section 3, we state the main result, and Section 4 compares it and the heterogeneous case.

2. Model

We consider a bottleneck model with finite numbers of commuters and time slots. Let $t = 1, ..., T$ be the available time slots for departure. Each time intervals can be every minute or every five minutes, for example. Let the set of time slots be $\mathcal{T} = \{1, ..., T\}$. At each time slot, c cars can go through a bottleneck, where c is a positive integer. If in the end of period $t-1$ the length of queue q_{t-1}, and if m_t cars arrives at the bottleneck in period t, then the length of the queue in the end of period t is given by $q_t = \max\{0, \tilde{q}_t\}$, where $\tilde{q}_0 \equiv 0$ and $\tilde{q}_t \equiv q_{t-1} + m_t - c$ for $t \geq 1$ is called the effective length of queue in period t. Effective and real queue length vectors are denoted $\tilde{q} = (\tilde{q}_1, ..., \tilde{q}_T)$ and $q = (q_1, ..., q_T)$, respectively. Let $i = 1, ..., n$ be commuters, and let the set of commuters be denoted by $N = \{1, ..., n\}$. Commuter i's choice (strategy) of departing time is denoted $\tau_i \in \mathcal{T}$. Given a strategy profile $\tau = (\tau_1, ..., \tau_n) \in \mathcal{T}^N$, and the resulting departure pattern is given by $m(\tau) = (m_1(\tau), ..., m_T(\tau)) \in \mathbb{Z}_+^{\mathcal{T}}$ and resulting effective and real queue length vectors are given by $\tilde{q}(\tau)$ and $q(\tau)$, respectively. Each commuter's payoff function is written as $u_i(t, q_t)$, where we assume that $u_i(t, q_t) > u_i(t, q_t + 1)$ for all $t \in \mathcal{T}$ and $q_t \in \mathbb{Z}_+$. That is, each commuter would prefer the situation of departing at time t with less congestion. We assume strict preferences that is generic case.

Definition 2.1 (Strict Preferences).
For all $i = 1, ..., n$, all $t, t' \in \mathcal{T}$ with $t \neq t'$, and all $q_t, q_{t'} \in \mathbb{Z}_+$, $u_i(t, q_t) \neq u_i(t', q_{t'})$.

Definition 2.2 (Nash Equilibrium).
A strategy profile τ is a Nash equilibrium if for all $i \in N$ and all $t \in \mathcal{T}$, $u_i(\tau, q_{\tau_i}(\tau)) \geq u_i(t, q_t(t, \tau_{-i}))$.

Before we give a characterization of Nash equilibria,we introduce new terms that is used for its characterization.

Definition 2.3 (Basin and Terrace).
1. A single slot t is said to be a **basin** at $\tau \in \mathcal{T}^N$ if $\tilde{q}_t(\tau) < 0$ and $\tilde{q}_{t-1}(\tau) \leq 0$.
2. A single slot t is a **single terrace** at $\tau \in \mathcal{T}^N$ if $\tilde{q}_t(\tau) = 0$ and $\tilde{q}_{t-1}(\tau) \leq 0$.
3. Consecutive slots $[t_1, t_2]$ with $1 \leq t_1 < t_2$ is a **connected terrace** at $\tau \in \mathcal{T}^N$ if $\tilde{q}_{t'}(\tau) > 0$ for all $t' \in [t_1, t_2)$ and $\tilde{q}_{t_2}(\tau) \leq 0$.

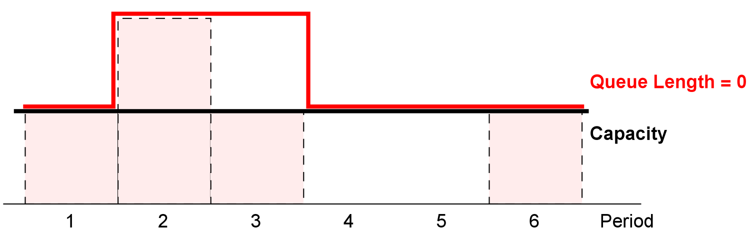

Figure 1: Basin and Terrace

Figure 1 is an example, where there are five commuters, six slots and $c = 1$, and the departure pattern is painted pink. In this example, $[2, 4]$ is a connected terrace while $t = 1$ and $t = 6$ each is a single terrace. In addition, $t = 6$ is a basin.

With these notions, the following is a characterization of Nash equilibria in this game.

Proposition 1.
A strategy profile τ is a Nash equilibrium if and only if for all $i \in N$,

1. $u_i(\tau_i, q_{\tau_i}(\tau)) \geq u_i(t', \max\{\tilde{q}_{t'}(\tau)+1, 0\})$ for all $t' < \tau_i$,
2. for $t' > \tau_i$,
   2a. $u_i(\tau_i, q_{\tau_i}(\tau)) \geq u_i(t', \max\{\tilde{q}_{t'}(\tau), 0\})$ for all $t' \in [t_1, t_2]$ with $t' > \tau_i$, where $[t_1, t_2]$ is a connected terrace at τ and $\tau_i \in [t_1, t_2]$.
   2b. Otherwise, $u_i(\tau_i, q_{\tau_i}(\tau)) \geq u_i(t', \max\{\tilde{q}_{t'}(\tau)+1, 0\})$.

Proof. First, suppose that profile τ is a Nash equilibrium. By definition, for all $i \in N$ and all $t' \in \mathcal{T}$,

$$u_i(\tau, q_{\tau_i}(\tau)) \geq u_i(t', q_{t'}(t', \tau_{-i})).$$

We show that for the both cases of $t' < \tau_i$ and $t' > \tau_i$, the queue-length at slot t', $q_{t'}(t', \tau_{-i})$ satisfies the above.

Note that profile (t', τ_{-i}) satisfies the following:

$$m_{\tau_i}(t', \tau_{-i}) = m_{\tau_i}(\tau) - 1, \tag{1}$$
$$m_{t'}(t', \tau_{-i}) = m_{t'}(\tau) + 1, \tag{2}$$
$$m_t(t', \tau_{-i}) = m_t(\tau) \quad \forall t \neq \tau_i, t'. \tag{3}$$

1. When $t' < \tau_i$:
   Since $q_{t'-1}(\tau) = q_{t'-1}(t', \tau_{-i})$ by (3), we obtain
$$\begin{aligned}\tilde{q}_{t'}(t', \tau_{-i}) &= q_{t'-1}(t', \tau_{-i}) + m_{t'}(t', \tau_{-i}) - c \\ &= q_{t'-1}(\tau) + (m_{t'}(\tau)+1) - c \\ &= (q_{t'-1}(\tau) + m_{t'}(\tau) - c) + 1 \\ &= \tilde{q}_{t'}(\tau) + 1.\end{aligned}$$
   Thus, $q_{t'}(t', \tau_{-i}) = \max\{\tilde{q}_{t'}(\tau)+1, 0\}$.
2. When $t' > \tau_i$:
   2a. When $t' \in [t_1, t_2]$, the connected terrace to which t' belongs:
   Note that since $\tilde{q}_{t''}(\tau) > 0$, so $\tilde{q}_{t''}(\tau) \geq 1$ for all $t'' \in [\tau_i+1, t'-1]$, $\tilde{q}_{t''}(t', \tau_{-i}) = \tilde{q}_{t''}(\tau) - 1 \geq 0$. It follows that
$$\begin{aligned}\tilde{q}_{t'}(t', \tau_{-i}) &= q_{t'-1}(t', \tau_{-i}) + m_{t'}(t', \tau_{-i}) - c \\ &= (q_{t'-1}(\tau) - 1) + (m_{t'}(\tau)+1) - c \\ &= q_{t'-1}(\tau) + m_{t'}(\tau) - c \\ &= \tilde{q}_{t'}(\tau).\end{aligned}$$
   Thus, $q_{t'}(t', \tau_{-i}) = \max\{\tilde{q}_{t'}(\tau), 0\}$.
   2b. Otherwise:
   We first show $q_{t''}(\tau) = q_{t''}(t', \tau_{-i})$ for some $t'' \in [\tau_i, t')$. Suppose not. Then, $q_t(\tau) > q_t(t', \tau_{-i}) \geq 0$ for all $t \in [\tau_i, t')$, implying that t' belongs to the same connected terrace as τ_i. This is a contradiction, since we have already considered in the case 2a..
   When $t'' = t'-1$, we immediately obtain $q_{t'-1}(\tau) = q_{t'-1}(t', \tau_{-i})$.
   When $t'' \neq t'-1$, using (3), we obtain
$$\begin{aligned}\tilde{q}_{t''+1}(t', \tau_{-i}) &= q_{t''}(t', \tau_{-i}) + m_{t''}(t', \tau_{-i}) - c \\ &= q_{t''}(\tau) + m_{t''}(\tau) - c \\ &= \tilde{q}_{t''+1}(\tau).\end{aligned}$$
   Thus, $q_{t''+1}(t', \tau_{-i}) = q_{t''+1}(\tau)$. Similarly,
$$\begin{aligned}q_{t''+2}(\tau) &= q_{t''+2}(t', \tau_{-i}), \\ &\vdots \\ q_{t'-1}(\tau) &= q_{t'-1}(t', \tau_{-i}).\end{aligned}$$
   With this assertion, we obtain
$$\begin{aligned}\tilde{q}_{t'}(t', \tau_{-i}) &= q_{t'-1}(t', \tau_{-i}) + m_{t'}(t', \tau_{-i}) - c \\ &= q_{t'-1}(\tau) + (m_{t'}(\tau)+1) - c \\ &= (q_{t'-1}(\tau) + m_{t'}(\tau) - c) + 1 \\ &= \tilde{q}_{t'}(\tau) + 1.\end{aligned}$$
   Thus, $q_{t'}(t', \tau_{-i}) = \max\{\tilde{q}_{t'}(\tau)+1, 0\}$.

Suppose next that profile τ satisfies the above but it is not a Nash equilibrium. Then, there exists some $i \in N$ who can improve by switching to another strategy t'. However, this implies that either case 1., case 2a., or case 2b. does not hold. This is a contradiction. ■

We also introduce a stronger concept, strong Nash equilibria. A strong Nash equilibrium is a strategy profile, which is immune to any coalitional deviation — that is, for any coalitional deviation, there is at least one player whose payoff does not improve by the deviation. Formally,

De nition 2.4 (Coalitional Deviation)**.**
A **coalitional deviation** (S, τ'_S) from profile τ is a pair of a nonempty subset of players $S \subseteq N$ and their strategy profile in S, $\tau'_S = (\tau'_i)_{i \in S} \in \mathcal{T}^S$ such that $u_i(\tau'_S, \tau_{-S}) > u_i(\tau)$ for all $i \in S$.

De nition 2.5 (Strong Nash Equilibrium)**.**
A profile τ is a **strong Nash equilibrium** if there is no coalitional deviation from τ.

Notice that a set of a strong Nash equilibrium is always a subset of a Nash equilibrium by definition.

3. The Analysis

In the following analysis, we assume homogeneous commuters. That is, we assume $u \equiv u_i$ for all $i \in N$. First, we show that the homogeneity in preferences is not enough to establish the general existence of Nash equlibria.

Example 1.
Consider the following three commuter problem: $N = \{1, 2, 3\}$, $\mathcal{T} = \{1, 2, 3\}$, $c = 1$, and players have the following preferences:

$$u(2,0) > u(1,0) > u(1,1) > u(3,0) > u(2,1) > u(1,2).$$

Then, there is no Nash equilibrium in this game.

Proof. There can be the following 9 strategy profiles.
1. $\tau^1 = (1,1,1)$ and $q(\tau^1) = (2,1,0)$. Then, player 1 moves to $t = 3$.
2. $\tau^2 = (1,1,2)$ and $q(\tau^2) = (1,1,0)$. Then, player 3 moves to $t = 3$.
3. $\tau^3 = (1,1,3)$ and $q(\tau^3) = (1,0,1)$. Then, player 2 moves to $t = 2$.
4. $\tau^4 = (1,2,2)$ and $q(\tau^4) = (0,1,0)$. Then, player 3 moves to $t = 3$.
5. $\tau^5 = (1,2,3)$ and $q(\tau^5) = (0,0,0)$. Then, player 3 moves to $t = 3$.
6. $\tau^6 = (2,2,2)$ and $q(\tau^6) = (0,2,1)$. Then, player 1 moves to $t = 1$.
7. $\tau^7 = (2,2,3)$ and $q(\tau^7) = (0,1,1)$. Then, player 1 moves to $t = 1$.
8. $\tau^8 = (2,3,3)$ and $q(\tau^8) = (0,0,1)$. Then, player 1 moves to $t = 1$.
9. $\tau^9 = (3,3,3)$ and $q(\tau^9) = (0,0,2)$. Then, player 1 moves to $t = 1$.

Hence, there is no Nash equilibrium. ■

Therefore, we need to restrict the domain of the preferences to establish the general existence of Nash equlibria.

However, we show a theorem which asserts that the set of Nash equlibria and the set of strong Nash equilibria coincide. Thus, it can be said that when a Nash equilibrium exists, it exhibits strong stability in the context of a coalitional deviation.

Theorem 3.1.
Assume that preferences are homogeneous. Then, the set of Nash equilibria coincides with the set of strong Nash equilibria.

Proof. The proof is omitted due to space constraint. ■

4. Heterogeneous Commuter Case

In this section, we analyze a more generalized case, commuters with heterogeneous preferences. We observe that there do not exist Nash equilibria in general, and also show that Proposition 3.1 is no longer true when assuming heterogeneous commuters.

Example 2.
Consider the following four commuter problem: $N = \{1, 2, 3, 4\}$, $\mathcal{T} = \{1, 2, 3, 4\}$, $c = 1$, and players have the following preferences:

$$u_1(1,2) > u_1(2,1) > u_1(3,1) > u_1(2,2) > u_1(1,3),$$
$$u_2(2,1) > u_2(1,2) > u_2(1,3) > u_2(2,2),$$

and for player $i = 3, 4$, $u_i(1,3) > u_i(t,0)$ for all $t = 2, 3, 4$.

Proof. In this example, players 3 and 4 always choose $t = 1$. Player 1 may choose $t = 1, 2, 3$, and player 2 may choose $t = 1, 2$. There can be the following 6 strategy profiles.

1. $\tau^1 = (1, 1, 1, 1)$ and $q(\tau^1) = (3, 2, 1, 0)$. In this case, player 1 moves to $t = 3$.
2. $\tau^2 = (2, 1, 1, 1)$ and $q(\tau^2) = (2, 2, 1, 0)$. In this case, player 1 moves to $t = 3$.
3. $\tau^3 = (3, 1, 1, 1)$ and $q(\tau^3) = (2, 1, 1, 0)$. In this case, player 2 moves to $t = 2$.
4. $\tau^4 = (1, 2, 1, 1)$ and $q(\tau^4) = (2, 2, 1, 0)$. In this case, player 2 moves to $t = 1$.
5. $\tau^5 = (2, 2, 1, 1)$ and $q(\tau^5) = (1, 2, 1, 0)$. In this case, player 2 moves to $t = 1$.
6. $\tau^6 = (3, 2, 1, 1)$ and $q(\tau^6) = (1, 1, 1, 0)$. In this case, player 1 moves to $t = 1$.

Hence, there is no Nash equilibrium in pure strategies. ■

The next example shows that when commuters' preferences differ from each other, the set of Nash equilibria may not coincide with the set of strong Nash equilibria.

Example 3.
Consider the following five commuter problem: $N = \{1, 2, 3, 4, 5\}$, $\mathcal{T} = \{1, 2, 3, 4, 5\}$, $c = 1$, and players have the following preferences:

$$\begin{aligned}
&u_1(1,1) > u_1(4,1) > u_1(1,2), \\
&u_2(2,0) > u_2(1,1) > u_2(1,2) > u_2(2,1), \\
&u_3(3,0) > u_3(1,2) > u_3(4,1) > u_3(3,1), \\
&u_4(1,4) > u_2(t,0) \qquad t = 2,3,4,5, \\
&u_5(4,4) > u_5(t,0) \qquad t = 1,2,3,5.
\end{aligned}$$

Proof. There is only one Nash equilibrium $\tau^* = (1, 1, 4, 1, 4)$ in this game. However, either $S = \{1, 3\}$ or $S = \{1, 2, 3\}$ can deviate from τ^* with $\tau'_{\{1,3\}} = (\tau'_1, \tau'_3) = (4, 1)$, or $\tau'_{\{1,2,3\}} = (\tau'_1, \tau'_2, \tau'_3) = (4, 2, 3)$, respectively. ■

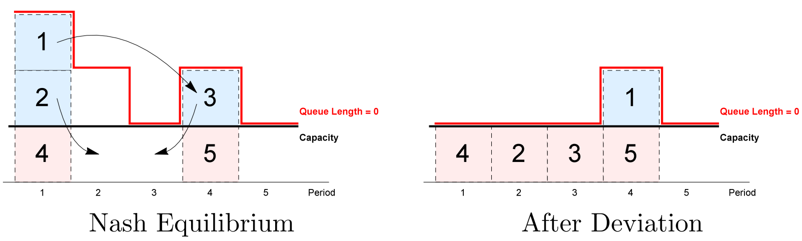

Figure 2: Deviation by $S = \{1, 2, 3\}$

5. Concluding Remarks

We have investigated the finite bottleneck game with homogeneous commuters. The main result of this paper is that the set of Nash equilibria coincides with that of strong Nash equilibria with this setting. In other words, the set of Nash equilibria "shrinks" to that of strong Nash equilibria, which may explain why the general existence of a Nash equilibrium is difficult to establish. We also have shown that this result cannot be applied to the heterogeneous commuter case.

In this regard, our model is quite contrasting to Milchtaich (1996) and Konishi et al. (1997a), where at least one Nash equilibrium always exists, and moreover the set of strong Nash equilibria is nonempty under fair conditions that characterize the “congestion (of each strategy)”—anonymity, independence from irrelevant choice and partial rivalry—even though preferences of players are heterogeneous.

As our examples show, we do not give the sufficient condition for the general existence of a Nash equilibrium. Hence, the remaining task is to establish such condition.

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